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# A FIRST APPROXIMATION OF THE QUASIPOTENTIAL IN PROBLEMS OF THE STABILITY OF SYSTEMS WITH RANDOM NON-DEGENERATE PERTURBATIONS†

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The problem of a local description (near to a stationary point or orbit) of the quasipotential—the Lyapunov function, used when analysing the stability of a system with small non-degenerate random perturbations is considered. First approximations are constructed for quasipotentials in neighbourhoods of these invariant sets. The quadratic forms specifying these approximations are governed by certain matrices. The construction of these matrices is reduced to the solution of Lyapunov matrix equations (which are algebraic in the case of stationary points, and differential with periodic coefficients in the case of orbits).

When investigating the stability of a stationary point $\xi$ of the deterministic system

$$dx = f(x)dt \tag{0.1}$$

acted upon by small random interference it is usual to change from Eq. (0.1) to the system of stochastic differential equations

$$dx = f(x)dt + \varepsilon\sigma(x)dw(t) \tag{0.2}$$

where the diffusion coefficient vanishes at the point $\xi$: $\sigma(\xi) = 0$, i.e. the point $\xi$ is also a stationary point for the stochastic system (0.2). It is for such systems, with noise which is degenerate in $\xi$, a theory has been developed for the usual types of stochastic stability: by probability, with unit probability, and with moments of different orders [1–3]. In a more realistic situation the system has non-degenerate noise capable of taking it out of the equilibrium position.

The number of papers investigating systems with non-degenerate noise $\sigma(\xi) \neq 0$ is much smaller. For similar systems the question of the existence of a stable stationary distribution has been investigated, and, in the linear case, one can fairly simply compute moments of various orders. For non-linear systems with non-degenerate low-intensity interference a method has been developed [4] for finding certain stochastic parameters (the mean departure time from a specified neighbourhood, the distribution function for the departure time, etc.) which enable one to judge the nature of the stability. Here a Lyapunov function of special form—a quasipotential (the minimum of the action functional)—is widely used. The quasipotential, being the Bellman function of some optimal control problem, satisfies the Hamilton–Jacobi equation, which is too difficult to solve when the original system is non-linear. Furthermore, in stability problems, one is, as a rule, mostly interested in the behaviour of the system in a small neighbourhood, and it is sufficient to have an appropriate local description of the quasipotential.

Section 1 of this paper investigates the quasipotential in the neighbourhood of the stationary point. Finding the matrix of the quadratic form—the first term of the expansion of the quasipotential—is reduced to the solution of Lyapunov's equation. An estimate of the error in finding the first approximation of the quasipotential is obtained.

Section 2 considers the stability problem for the case when one takes a closed curve—an orbit—to be the limiting variant set of the original deterministic system (0.1), instead of a point. Necessary and sufficient conditions for exponential orbital stability (EOS), associated with the Andronov–Witt theorem and its analogues, are related to the first Lyapunov method. Orbital Lyapunov functions were introduced in [7] to investigate the EOS of deterministic systems. Using these functions criteria were obtained in [8] for the mean-square stability of systems that were degenerate along the orbit. For non-degenerate noise the quasipotential is taken to be the orbital Lyapunov function. As in Section 1, the first term of the expansion of the quasipotential in the neighbourhood of the orbit is given by some matrix.

In the given case this matrix is a periodic function of time. Its construction reduces to the solution of a Lyapunov matrix equation with periodic coefficients. The existence and uniqueness of this solution is investigated.

In Section 3 a method is proposed for the two-dimensional (plane orbit) case that enables one to obtain the first approximation of the quasipotential in analytic form. Section 4 contains examples illustrating the results obtained, both in the case of a stationary point and for an orbit.

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## 1. THE QUASIPOTENTIAL IN THE NEIGHBOURHOOD OF A STATIONARY POINT

Suppose that in Eqs (0.1) and (0.2) $x$ is an $n$-dimensional vector, $w(t)$ is an $m$-dimensional standard Wiener process, and the sufficiently smooth functions $f(x)$ and $\sigma(x)$ have dimensions of $n$ and $n \times m$. We shall assume that in system (0.1) $\xi$ is an exponentially stable isolated stationary point ($f(\xi) = 0$) and that a neighbourhood $U$ of the point $\xi$ exists in which the noise of system (0.2) is non-degenerate, i.e. for any $x \in U$ the matrix $S(x) = \sigma(x)\sigma^T(x)$ is positive-definite (and, in particular, $\sigma(\xi) \neq 0$ and $S(\xi) > 0$). One can, of course, choose $\xi$ to be at the origin of coordinates. However, we will not do this because we wish to use the same notation in the orbital case. We introduce the system

$$y^{\cdot} = -f(y) + u, \quad y(0) = x \tag{1.1}$$

where $y$ is an $n$-dimensional state and $u$ is an $n$-dimensional control, taking the solution of system (1.1) from the initial position $x$ to the point $\xi$

$$\lim_{s\to\infty} y(s) = \xi \tag{1.2}$$

The quasipotential [4] for system (0.2) in the neighbourhood $U$ of the stationary point $\xi$ is the function

$$\varphi(x) = \inf_u \frac{1}{2}\int_0^\infty u^T(s)Q(y(s))u(s)ds, \quad Q(y) = S^{-1}(y) \tag{1.3}$$

where the functions $u(s)$ and $y(s)$ satisfy system (1.1), (1.2). For those $y(s)$ for which the matrix $S(y(s))$ degenerates (which can happen if $y(s)$ does not belong to $U$), we make the integrand in (1.3) formally equal to plus infinity. We write

$$\frac{\partial\varphi}{\partial x} = \left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_n}\right)^T, \quad \frac{\partial^2\varphi}{\partial x^2} = \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right)_{i,j=1}^n$$

In the neighbourhood $U$ the function $\varphi(x)$ satisfies the Hamilton–Jacobi equation

$$\left(f(x), \frac{\partial\varphi}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}, S(x)\frac{\partial\varphi}{\partial x}\right) = 0 \tag{1.4}$$

Here

$$\varphi(\xi) = 0, \quad \varphi(x) \geqslant 0 \tag{1.5}$$

We construct the first approximation of the quasipotential in a neighbourhood of the stationary point. We assume that the function $\varphi(x)$ is sufficiently smooth in the neighbourhood $U$. We can then write

$$\varphi(x) = \varphi(\xi) + \left(x - \xi, \frac{\partial\varphi}{\partial x}(\xi)\right) + \frac{1}{2}\left(x - \xi, \frac{\partial^2\varphi}{\partial x^2}(\xi)(x - \xi)\right) + O(|x - \xi|^3) \tag{1.6}$$

The equality $\partial\varphi/\partial x(\xi) = 0$ follows from (1.5), i.e. the first two terms of the expansion (1.6) are zero. As a result we have the representation

$$\varphi(x) = \varphi_1(x) + O(|x - \xi|^3) \tag{1.7}$$

where the quadratic form $\varphi_1(x) = (x - \xi)^T V(x - \xi)$ is defined by the matrix

$$V = \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}(\xi) \tag{1.8}$$

After differentiating (1.4) twice with respect to $x$ and substituting $x = \xi$ we obtain the equation

$$F^T V + VF + 2VSV = 0 \tag{1.9}$$

$$F = \left( \frac{\partial f_i}{\partial x_j}(\xi) \right)_{i,j=1}^{n}, \quad S = S(\xi) = \sigma(\xi)\sigma^T(\xi)$$

for the matrix $V$.

Due to the exponential stability of the point $\xi$ in system (0.1), $F$ is a Hurwitz matrix. If $V$ is a positive definite matrix, then the matrix $W = V^{-1}$ satisfies Lyapunov's equation

$$FW + WF^T + 2S = 0 \tag{1.10}$$

which is obtained after multiplying the left-hand side of Eq. (1.9) on the left and right by $V^{-1}$.

Thus, assuming the smoothness of $\varphi(x)$ and the positive definiteness of matrix (1.8), we obtain relation (1.7) where the matrix $V$ of the quadratic form $\varphi_1(x)$ can be found by inverting the matrix $W$, which is the solution to Eq. (1.10).

We now consider Eq. (1.10) to be the initial equation. Because $F$ is a Hurwitz matrix and $S$ is positive definite, Eq. (1.10) has a unique solution: the positive-definite matrix $W$. Then $V = W^{-1}$ satisfies Eq. (1.9). We consider the function $\psi(x) = (x - \xi)^T V(x - \xi)$. We note that $\psi(x)$ is the exact quasipotential for a linear system with additive noise

$$dz = Fzdt + \varepsilon\sigma(\xi)dw(t), \quad z = x - \xi$$

which is the first-approximation system for (0.2) near the point $\xi$.

*Lemma* 1. For system (1.1) with control

$$u_*(y) = -S(y)\frac{\partial \psi}{\partial x}(y)$$

the solution $y(t) \equiv \xi$ is exponentially stable.

*Proof.* The right-hand side of system (1.1) can be written in the form

$$-f(y) + u_*(y) = -F(y-\xi) - 2SV(y-\xi) + O(|y-\xi|^3) = F_*(y-\xi) + O(|y-\xi|^3)$$

where $F_* = -F - 2SV$ is the matrix of the first-approximation system for system (1.1) with control $u_*(y)$. Since

$$F_*^T V + VF_* = -2VSV$$

where the matrices $V$ and $VSV$ are positive definite, $F_*$ is a Hurwitz matrix. The control $u_*(y)$ therefore stabilizes system (1.1).

*Theorem* 1. Suppose $\psi(x) = (x - \xi)^T V(x - \xi)$, where $V = W^{-1}$, and $W$ is a solution of Eq. (1.10). We have the representation

$$\varphi(x) = \psi(x) + O(|x-\xi|^3)$$

for the quasipotential (1.3) in the neighbourhood $U$ of the point $\xi$.

*Proof.* Consider the functional $L(x, u) = J(x, u) - \psi(x)$ where

$$J(x,u) = \frac{1}{2}\int_0^\infty u^T(s)Q(y(s))u(s)ds$$

is computed along the solution $y(s)$ of system (1.1) with control $u(s)$. By Lemma 1, $J(x, u_*) < \infty$. Using the technique of [9], we can write

$$L(x,u)=\int_0^\infty G(y(t),u(t))dt$$

$$G(y,u)=\frac{1}{2}u^TQ(y)u+\left(\frac{\partial\psi}{\partial x}(y),-f(y)+u\right)$$

For any fixed $y$ the function $G(y, u)$ is a second-order form in $u$, such that for $u_*=-Q^{-1}(y)(\partial\psi/\partial x)(y)$ we have the relations

$$G(y,u)\geqslant G(y,u_*)=-2\left[(y-\xi)^TVQ^{-1}(y)V(y-\xi)+(y-\xi)^TVf(y)\right]$$

Since

$$Q^{-1}(y)=\sigma(y)\sigma^T(y)=S+O(|y-\xi|)$$

$$f(y)=f(\xi)+F(y-\xi)+O(|y-\xi|^2)$$

we have

$$G(y,u_*)=-(y-\xi)^T[2VSV+VF+F^TV](y-\xi)+O(|y-\xi|^3)$$

Using (1.9), we obtain $G(y, u_*) = O(|y-\xi|^3)$. Thus $L(x, u) \geqslant -|O(|x-\xi|^3)|$ for any $u$, from which it follows that $\varphi(x) \geqslant \psi(x) - O(|y-\xi|^3)$. On the other hand

$$L(x,u_*)=\int_0^\infty G(y,u_*)dt=O(|x-\xi|^3)$$

i.e. $J(x, u_*) - \psi(x) = O(|x-\xi|^3)$. Since $\varphi(x) \leqslant J(x, u)$, we have $\varphi(x) \leqslant \psi(x) + |O(|x-\xi|^3)|$.

## 2. THE QUASIPOTENTIAL IN A NEIGHBOURHOOD OF AN ORBIT

Suppose $x = \xi(t)$ is a $T$-periodic solution of the deterministic system (0.1), which is not a stationary point ($f(\xi(t)) \neq 0$), satisfying some initial condition $\xi(0) = \xi_0$, and $\Gamma$ is the phase trajectory of this solution (orbit). It is assumed that a neighbourhood $U$ of the orbit $\Gamma$ exists such that $f(x)$ and $\sigma(x)$ are sufficiently smooth functions in $U$, the matrix $\sigma(x)^T\sigma(x)$ is positive definite for every point $x \in U$, including $x \in \Gamma$ (the noise of system (0.2) is non-degenerate on the orbit), and that for any point $x \in U$ there is a unique point $\gamma(x) \in \Gamma$ which is the point on the trajectory $\Gamma$ that is nearest to $x$. Here the vector $\Delta(x) = x - \gamma(x)$ is the displacement of the point $x$ from the orbit orthogonal to the tangent vector $f(\gamma(x))$. We assume that the solution $x = \xi(t)$ of system (0.1) is exponentially orbitally stable.

The quasipotential of system (0.2) in the neighbourhood $U$ of the orbit $\Gamma$ is the function [4]

$$\varphi(x)=\inf_u\frac{1}{2}\int_0^\infty u^T(x)Q(y(s))u(s)ds,\quad Q(y)=\left[\sigma(y)\sigma^T(y)\right]^{-1} \tag{2.1}$$

where $u$ is an $n$-dimensional control taking the system

$$y^{\cdot}=-f(y)+u \tag{2.2}$$

from the initial position $y(0) = x$ onto the orbit $\Gamma$

$$\lim_{s\to\infty}\Delta(y(s))=0$$

Assuming that it is sufficiently smooth, the function $\varphi(x) \geqslant 0$ satisfies the Hamilton–Jacobi equation in the neighbourhood $U$ together with conditions on $\Gamma$

$$\left(f(x),\ \frac{\partial\varphi}{\partial x}\right)+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x},\ Q^{-1}(x)\ \frac{\partial\varphi}{\partial x}\right)=0$$

$$\varphi|_\Gamma = 0, \quad \left.\frac{\partial\varphi}{\partial x}\right|_\Gamma = 0 \tag{2.3}$$

We construct the first approximation of the quasipotential in the neighbourhood $U$ of the orbit $\Gamma$. In a neighbourhood of the point $\varphi \in \Gamma$ we can write

$$\varphi(x) = \varphi(\gamma) + (x-\gamma, \frac{\partial\varphi}{\partial x}(\gamma)) +$$
$$+\frac{1}{2}(x-\gamma, \frac{\partial^2\varphi}{\partial x^2}(\gamma)(x-\gamma)) + O(|x-\gamma|^3) \tag{2.4}$$

For each fixed $x$ in (2.4) it is natural to take $\gamma = \gamma(x)$. We put $\Phi(x) = ½(\partial^2\varphi/\partial x^2)(\gamma(x))$. It follows from (2.4) that $\varphi(x) = \varphi_1(x) + O(|\Delta(x)|^3)$, where it is natural to call the function $\varphi_1(x) = \Delta^T(x)\Phi(x)\Delta(x)$ the orbital quadratic form.

We introduce the function $\eta(t) = \xi(T-t)$ which is the solution of Eq. (2.2) when $u \equiv 0$. We denote by $t(\eta)$ the inverse function to $\eta(t)$ in the interval $[0, T)$. Then $\vartheta(x) = t(\gamma(x))$ is the time at which the solution $\eta(t)$ is at the point $\gamma(x)$ of the trajectory $\Gamma$, i.e. $\eta(t(\gamma(x))) = \gamma(x)$. Here we have the representation

$$\Phi(x) = V(t(\gamma(x))), \quad V(t) = \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}(\eta(t))$$

The function $\Phi(x)$ is thus uniquely defined in the neighbourhood $U$ by the solution $\eta(t)$ and the $T$-periodic matrix $V(t)$. This simple representation of the matrix $\Phi(x)$ is due to the fact that it takes the same value at all points of the neighbourhood $U$ lying in the $(n-1)$-dimensional plane orthogonal to the orbit $\Gamma$ at the point $\gamma(x)$. For $\varphi_1(x)$ we have the representation

$$\varphi_1(x) = (x - \eta(\vartheta(x)))^T V(\vartheta(x))(x - \eta(\vartheta(x)))$$

Doubly differentiating the Hamilton–Jacobi equation (2.3) with respect to $x$ and substituting $x = \eta(t)$, we obtain for $V(t)$ (see [7]) the matrix Bernoulli equation

$$V^{\cdot} - F^T V - VF - 2VSV = 0 \tag{2.5}$$

Here

$$F(t) = \left(\frac{\partial f_i}{\partial x_j}(\eta(t))\right)_{i,j=1}^n, \quad S(t) = \sigma(\eta(t))\sigma^T(\eta(t))$$

are $T$-periodic matrices.

Differentiating the identity $(\partial\varphi/\partial x)(\eta(t)) \equiv 0$ with respect to $t$, we obtain

$$V(t)f(\eta(t)) \equiv 0 \tag{2.6}$$

i.e. the matrix $V(t)$ is degenerate for all $t$.

We denote by $P_f$ the matrix corresponding to the operator of projection onto the subspace orthogonal to the vector $f \neq 0$, $P_f = E - (ff^T/f^Tf)$. We introduce the $T$-periodic matrix $P(t) = P_{f(\eta(t))}$. At any time $t$ the matrix $P(t)$ specifies the operator of projection onto the subspace orthogonal to the orbit $\Gamma$ at the point $\eta(t)$.

*Definition* [7]. A $T$-periodic symmetric matrix $A(t)$ is called $P(t)$-positive-definite at time $t$ if for any vector $z$ such that $P(t)z \neq 0$ the inequality $z^TA(t)z > 0$ holds. A matrix $A(t)$ which is $P(t)$-positive-definite for all $0 \leq t < T$ is called $P$-positive-definite and we can write $A \overset{P}{>} 0$.

We will denote by $\Sigma$ the space of continuous $T$-periodic symmetric $(n \times n)$-matrices $B(t)$ such that for any $t \in [0, T)$ the equality $B(t)f(\eta(t)) = 0$ holds. For such matrices we have the identity

$$P(t)B(t) \equiv B(t)P(t) \equiv B(t)$$

We consider in $\Sigma$ the cone of matrices $K = \{A \in \Sigma | A \overset{P}{>} 0\}$.

The construction of the first approximation of the quasipotential in the neighbourhood of the orbit $\Gamma$ reduces, by Theorems 2–4 (see below), to the solution of Lyapunov's equation

$$W' + FW + WF' + 2PSP = 0 \tag{2.7}$$

*Theorem* 2. When the solution $\xi(t)$ of system (0.1) is EOS, Eq. (2.7) has a unique solution $W(t)$ in $K$.

*Proof.* We relate the matrices

$$F_1(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad P_1(t) = P_{f(\xi(t))}$$

to the solution $\xi(t)$ of system (0.1).

We denote by $\Sigma_1$ the space of continuous symmetric $T$-periodic $(n \times n)$-matrices $V(t)$ such that for any $t \in [0, T)$ we have $V(t)f(\xi(t)) = 0$. In $\Sigma_1$ we consider the cone

$$K_1 = \left\{ V \in \Sigma_1 | V \overset{P_1}{>} 0 \right\}$$

and the scalar product

$$\langle V, W \rangle = \int_0^T \mathrm{tr}(V(t)W(t))dt$$

We consider the operator $\Lambda$ defined on the continuously-differentiable matrices $V \in \Sigma_1$ by

$$\Lambda(V) = V' + F_1^T V + VF_1$$

Due to the EOS of the solution $\xi(t)$ (see [7]), for any element $C \in K_1$ the equation

$$\Lambda(V) = -C$$

has a unique solution $V \in K_1$. Because $K_1$ is the generating cone [10] in $\Sigma_1$, the operator is invertible throughout the space $\Sigma_1$. The operator $\Lambda^*$, conjugate to $\Lambda$, is given on $D \in \Sigma_1$ by

$$\Lambda^*(D) = -D' + F_1 D + DF_1^T$$

It can be shown that for any element $G \in K_1$, the element $D = -(\Lambda^{-1})^*(G)$ is the unique solution in $K_1$ of the equation

$$-D' + F_1 D + DF_1^T = -G \tag{2.8}$$

Making the change of variable $t = T - \tau$ in (2.8) we obtain relation (2.7) for the matrices

$$F(\tau) = F_1(T - \tau), \quad W(\tau) = D(T - \tau) \in K$$

$$2P(\tau)S(\tau)P(\tau) = G(T - \tau) \in K$$

Theorem 2 is proved.

*Theorem* 3. Let $W(t) \in K$ be a solution of Eq. (2.7). Then $V(t) = W^+(t) \in K$ and the equation

$$V' - F^T V - VF - 2VSV = 0 \tag{2.9}$$

is satisfied.

*Proof.* The inclusion of $V$ in the cone $K$ is a consequence of the fact that $W \in K$. Since $V = VWV = VP = PV$, we have

$$VW^{\cdot}V = -V^{\cdot} + V^{\cdot}(E-P) + (E-P)V^{\cdot} \tag{2.10}$$

Multiplying the left-hand side of (2.7) on the left and right by $V$, we obtain, taking (2.10) into account, the relation

$$-V^{\cdot} + V^{\cdot}(E-P) + (E-P)V^{\cdot} + VFP + PF^TV + 2VSV = 0 \tag{2.11}$$

Differentiating the identity $V(t)f(\eta(t)) \equiv 0$ with respect to $t$, we obtain

$$V^{\cdot}f(\eta(t)) - VFf(\eta(t)) \equiv 0$$

whence it follows that $V^{\cdot}(E-P) = VF(E-P)$, and consequently, relation (2.11) acquires the form

$$-V^{\cdot} + VF + F^TV + 2VSV = 0$$

Relation (2.9) therefore holds for the matrix $V = W^+$.
Let $\psi(x) = \Delta^T(x)V(\vartheta(x))\Delta(x)$.

*Lemma* 2. For system (2.2) with control $u^* = -\sigma(y)\sigma^T(y)(\partial\psi/\partial x)(y)$ the solution $y = \eta(t)$ is exponentially orbitally stable.

*Proof*. For $f_*(y) = -f(y) + u_*(y)$ we have the relation

$$F_*(t) = \frac{\partial f_*}{\partial x}(\eta(t)) = -F(t) - 2S(t)V(t)$$

From (2.9) it follows that

$$V^{\cdot} + F_*^TV + VF_* + 2VSV = 0$$

where the matrix $VSV \in K$. The EOS of solution $y = \eta(t)$ follows from the criteria of [7].

*Theorem* 4. Let $\psi(x) = \Delta^T(x)V(\vartheta(x))\Delta(x)$, where $V(t) = W^+(t)$, and $W(t) \in K$ is a solution of Eq. (2.7). Then the representation

$$\varphi(x) = \psi(x) + O(|\Delta(x)|^3)$$

holds for the quasipotential (2.1) in the neighbourhood of the orbit $\Gamma$.

*Proof*. Consider the function $L(x,u) = J(x,u) - \psi(x)$ where

$$J(x,u) = \frac{1}{2}\int_0^\infty u^T(x)Q(y(s))u(s)ds$$

is calculated with $y$ and $z$ from (2.2). Using the technique of [9], we can write

$$L(x,u) = \int_0^\infty G(y,u)ds, \quad G(y,u) = \frac{1}{2}u'Q(y)u + \left(\frac{\partial\psi}{\partial x}(y),\ -f(y)+u\right)$$

Let $u_* = -Q^{-1}(y)(\partial\psi/\partial x)(y)$. Then for any $y$ and $u$ the following relation holds

$$G(y,u) \geqslant G(y,u_*) = \frac{1}{2}\left(\frac{\partial\psi}{\partial x}(y),\ Q^{-1}(y)\frac{\partial\psi}{\partial x}(y)\right) + \left(\frac{\partial\psi}{\partial x}(y), f_*(y)\right)$$

Using the expansion lemma from [7], we obtain

$$\left(\frac{\partial\psi}{\partial x}(y), f_*(y)\right) = (y-\eta(t))^T\left[F_*^TV + VF_* + V^{\cdot}\right](y-\eta(t)) + O(|y-\eta(t)|^3)$$

For the remaining terms of $G(y, u_*)$ we have the equality

$$\frac{\partial \psi_*}{\partial x}(y) = 2V(t)(y-\eta(t)) + O(|y-\eta(t)|^3)$$

$$Q^{-1}(y) = \sigma(y)\sigma'(y) = S(t) + O(|y-\eta(t)|)$$

The relations

$$G(y,u_*) = O(|\Delta(y)|^3), \quad G(y,u_*) \geq -|O(|\Delta(y)|^3)|$$

follow from these expansions and from (2.12).

Because $u_*$ stabilizes system (2.2) (see Lemma 2), $L(x, u_*) \geq -|O(|\Delta(x)|^3)|$. It follows from this inequality that

$$\varphi(x) = \min_u J(x,u) \geq \psi(x) - |O(|\Delta(x)|^3)|$$

On the other hand

$$L(x,u_*) = \int_0^\infty G(y,u_*)dt = O(|\Delta(x)|^3)$$

i.e. $J(x, u_*) = \psi(x) + O(|\Delta(x)|^3)$. Since $\varphi(x) \leq J(x, u)$, we have

$$\varphi(x) \leq \psi(x) + |O(|\Delta(x)|^3)|$$

Finally, we obtain the inequality

$$-|O(|\Delta(x)|^3)| \leq \varphi(x) - \psi(x) \leq |O(|\Delta(x)|^3)|$$

Theorem 4 is proved.

*Remark.* Note that the assumption of the smoothness of the quasipotential $\varphi(x)$ was only used in preliminary arguments of a heuristic nature when deriving Eqs (2.5) and (2.6). This assumption was not used to obtain the precise results.

## 3. THE PLANE ORBIT CASE

We consider the Lyapunov equation

$$W^{\cdot} + FW + WF^T + 2PSP = 0 \tag{3.1}$$

for $n = 2$. In this case the projection matrix is of rank 1 and is given by $P(t) = \upsilon(t)^T\upsilon(t)$ where $\upsilon(t)$, the normalized eigenvector of the matrix $P(t)$, is orthogonal to $f(\eta(t))$ for all $t$. It can be shown that

$$W(t) = \mu(t)\upsilon(t)\upsilon^T(t), \quad P(t)S(t)P(t) = \beta(t)\upsilon(t)\upsilon^T(t) \tag{3.2}$$

where $\mu(t) > 0$, $\beta(t) > 0$ are $T$-periodic functions which are the eigenvalues of the matrices $S$ and $PSP$ for all $t$. Note that $\beta(t) = \upsilon^T(t)S(t)\upsilon(t)$. Substituting (3.2) into (3.1), we obtain

$$\mu^{\cdot}\upsilon\upsilon^T + \mu\upsilon^{\cdot}\upsilon^T + \mu\upsilon(\upsilon^T)^{\cdot} + \mu[\upsilon\upsilon^T F^T + F\upsilon\upsilon^T] + 2\beta\upsilon\upsilon^T = 0$$

After multiplying on the left by $\upsilon^T$ and on the right by $\upsilon$ an taking $\upsilon^T\upsilon = 1$ into account, we obtain

$$\mu^{\cdot} + \mu[\upsilon^T\upsilon^{\cdot} + (\upsilon^T)^{\cdot}\upsilon] + \mu\upsilon^T[F^T + F]\upsilon + 2\beta = 0$$

Since $\upsilon^T\upsilon^{\cdot} + (\upsilon^T)^{\cdot}\upsilon = [\upsilon^T\upsilon]' \equiv 0$, we have

$$\mu^{\cdot} + \upsilon^T[F^T + F]\upsilon \cdot \mu + 2\beta = 0$$

We have thus obtained a linear equation for $\mu$. The required $T$-periodic solution ($\mu(T) = \mu(0)$) is found analytically.

Using the constraint $V = W^+$, we obtain the matrix $V(t) = \mu^{-1}(t)\upsilon(t)\upsilon^T(t)$. Hence the function giving the first approximation of the quasipotential has the form $\psi(x) = \mu^{-1}(\vartheta(x))|\Delta(x)|^2$. The function $\mu(t)$ is a simple numerical characteristic of the stability of the system in a neighbourhood of the orbit $\Gamma$. For example, the points where the trajectory is most likely to leave the neighbourhood $|\Delta(x)| \leq \rho$ for small $\rho$ lie near that section of the orbit $\Gamma$ where $\mu(t)$ takes its largest values.

## EXAMPLES

1. Consider the system

$$x_1^{\cdot} = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2) + \varepsilon w_1^{\cdot}$$

$$x_2^{\cdot} = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) + \varepsilon w_2^{\cdot}$$

with small additive noise. Introducing polar coordinates $x_1 = r\cos\varphi, x_2 = r\sin\varphi$ we obtain

$$r^{\cdot} = \alpha r - r^3 + \frac{\varepsilon^2}{2r} + \varepsilon w^{\cdot}, \quad w^{\cdot} = (x_1 w_1^{\cdot} + x_2 w_2^{\cdot})/r$$

$$\varphi^{\cdot} = 1 + \frac{\varepsilon}{r}\nu^{\cdot}, \quad \nu^{\cdot} = (x_1 w_2^{\cdot} - x_2 w_1^{\cdot})/r$$

It is well known that in the deterministic case ($\varepsilon = 0$), when the parameter $\alpha$ crosses zero from left to right, the asymptotic stability of the stationary point $r_0 = 0$ turns into instability for $\alpha > 0$. Here an exponentially stable orbit appears—a circle of radius $r_0 = \sqrt{(\alpha)}$ (soft Andronov–Hopf bifurcation). It is interesting to trace how this qualitative transition (stationary point to cycle) is accompanied by quantitative changes in the asymptotic behaviour [4] $\kappa = \lim_{\varepsilon\to 0}\varepsilon^2 \ln E\{\tau^\varepsilon\}$ of the mean time of departure $\tau^\varepsilon$ of the trajectory from the neighbourhood $U$ under the action of small random perturbations.

In the stochastic case ($\varepsilon \neq 0$) we find:

(a) for $\alpha < 0$ the first approximation of the quasipotential in the neighbourhood $U = \{(x_1, x_2) | r \leq \rho\}$—a disc with diameter $2\rho$ and centre at the stationary point $r_0 = 0$, is the function $\psi = -\alpha(r - r_0)^2 = -\alpha r^2$, and $\kappa = -\alpha\rho^2 + O(\rho^3)$;

(b) for $\alpha > 0$ the first approximation of the quasipotential in the neighbourhood $U = (x_1, x_2) | |r - r_0| \leq \rho, r_0 = \sqrt{(\alpha)}\}$—an annulus of width $2\rho$—is the function $\psi = 2\alpha(r - r_0)^2$; and here $\kappa = 2\alpha\rho^2 + O(\rho^3)$.

2. Consider the van der Pol equation with small non-degenerate noises

$$x_1^{\cdot} = x_2 + \varepsilon\sigma_1(x_1, x_2)w_1^{\cdot}$$

$$x_2^{\cdot} = -x_1 + \alpha x_2(1 - x_1^2) + \varepsilon\sigma_2(x_1, x_2)w_2^{\cdot}$$

We know that the asymptotically stable orbit $\Gamma_\alpha$ of the solution $x = \xi(t, \alpha)$ of the deterministic van der Pol equation for small $\alpha > 0$ differs little from a circle of radius 2. Here the period $T_\alpha$ of the solution $\xi(t, \alpha)$ is close to $2\pi$. The first approximation of the quasipotential in a small neighbourhood of the orbit $\Gamma_\alpha$ is the function

$$\psi(x, \alpha) = |\Delta(x, \alpha)|^2 / \mu(\vartheta(x, \alpha), \alpha)$$

Here $\Delta(x, \alpha)$ is the deviation of the point $x$ from the orbit $\Gamma_\alpha$, the function $\mu(t, \alpha)$ is a $T_\alpha$-periodic solution of the equation

$$\mu^{\cdot} + a(t, \alpha)\mu + b(t, \alpha) = 0$$

and the $T_\alpha$-periodic coefficients have the expansions

$$a(t, \alpha) = \alpha(-2 + 3\cos 4t - \cos 2t) + O(\alpha^2)$$

$$b(t, \alpha) = b_0(t) + O(\alpha)$$

$$b_0(t) = \cos^2 t\sigma_1^2(2\cos t,\ 2\sin t) + \sin^2 t\sigma_2^2(2\cos t, 2\sin t)$$

The corresponding expansion of the function $\alpha\mu(t, \alpha)$ has the form

$$\alpha\mu(t, \alpha) = \frac{c}{4\pi} + O(\alpha), \quad c = \int_0^{2\pi} b_0(t)\,dt$$

As a result we obtain

$$\psi(x,\alpha) = 4\pi c^{-1}\alpha|\Delta(x,\alpha)|^2(1+O(\alpha))$$

The asymptotic behaviour $\kappa = \lim_{\varepsilon\to 0} \varepsilon^2 \ln E\{\tau^\varepsilon\}$ of the time of departure of $\tau^\varepsilon$ from the neighbourhood $U_\rho = \{x \| \Delta(x, \alpha)| \leqslant \rho\}$ of the orbit $T_\alpha$ for small $\alpha$ and $\rho$ is as follows: $\kappa \approx 4\pi c^{-1}\alpha\rho^2$.

*Remark.* We made mistakes in Theorems 2 and 3 in [8]. These can be corrected if, for example, in the inequality $C(\tau) - \alpha(\tau)I \overset{P_T}{\geqq} 0$ in Theorem 2 $I$ is replaced by $V(\tau)$, and in Theorem 3 the condition

$$\frac{1}{T}\int_0^T \mu(\tau)\,\mathrm{tr}V(\tau)d\tau < 1$$

is replaced by the condition

$$\max_{0\leqslant\tau\leqslant T}[\mu(\tau)\,\mathrm{tr}\,V(\tau)] < 1$$

## REFERENCES


1. KATS I. Ya. and KRASOVSKII N. N., The stability of systems with random parameters. *Prikl. Mat. Mekh.* **24**, 5, 809–823, 1960.
2. KHAS'MINSKII R. Z., *Stability of Systems of Differential Equations When There are Random Perturbations of their Parameters*. Nauka, Moscow, 1969.
3. KUSHNER G. J., *Stochastic Stability and Control*. Mir, Moscow, 1969.
4. VENTSEL' A. D. and FREIDLIN M. I., *Fluctuations in Dynamical Systems under the Action of Small Random Perturbations*. Nauka, Moscow, 1979.
5. ANDRONOV A. A. and WITT A. A., On Lyapunov stability. *Zh. Eks. Teor. Fiz.* **2**, 5, 373–374, 1933.
6. HARTMAN G., *Ordinary Differential Equations*. Mir, Moscow, 1970.
7. MIL'SHTEIN G. N., Stability and stabilization of periodic motions of autonomous systems. *Prikl. Mat. Mekh.* **41**, 4, 744–749, 1977.
7. MIL'SHTEIN G. N. and RYASHKO L. B., Stability and stabilization of orbits of autonomous systems under random perturbations. *Prikl. Mat. Mekh.* **56**, 6, 951–958, 1992.
9. MIL'SHTEIN G. N., The use of successive approximations to solve an optimal problem. *Avtom. Telemekh.* **25**, 3, 321–329, 1964.
10. KRASNOSEL'SKII M. A., LIFSHITS E. A. and SOBOLEV A. V., *Positive Linear Systems*. Nauka, Moscow, 1985.


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